LOCAL SYSTEMS OF SHAPOVALOV ELEMENTS

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ABSTRACT. Shapovalov introduced elements in the enveloping algebra of a semisimple Lie algebra that when applied to the highest weight vector of a Verma module generates a Verma submodule. A Shapovalov element induces a containment of Verma modules uniformly across all highest weights in a particular root hyperplane. As a consequence of the Jantzen conjecture, the generator of a Verma submodule is known to lie in a definite level of the Jantzen filtration of the Verma module. Is it possible to choose a Shapovalov element that will place itself naturally into the correct filtration level uniformly across the root hyperplane? This paper considers this problem in the case of $\mathfrak{sl}(n)$. There are weights, called singular weights, where such Shapovalov elements cannot exist even in a local sense. What then can be said for the non-singular weights? A system of elements is introduced using Carter’s theory of lowering operators [1]. In the examples, given any non-singular weight, there is an element of this system that provides a local solution to the problem. It is conjectured that this description extends to the general case and characterizes the singular weights.

0. Notation

Let $\mathfrak{g} = \mathfrak{sl}(n, K)$, where $K$ is an algebraically closed field of characteristic 0. Take $\mathfrak{h}$ to be the diagonal subalgebra and let $\mathfrak{n}$ be the algebra of lower triangular matrices. $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let $\mathfrak{n}^+ = \mathfrak{n}$. A vector $v$ in a $\mathfrak{g}$-module is a primitive vector if $\mathfrak{n}^+, v = 0$.

Let $I = \{(k, m) : 1 \leq k < m \leq n\}$ and let $S = \{(k, k + 1) : 1 \leq k \leq n - 1\}$. If $(k, m) \in I$ define $e_{km} = E_{km}$, $f_{km} = E_{mk}$, and $h_{km} = E_{kk} - E_{mm}$. If $\lambda \in \mathfrak{h}^*$, let $\lambda_k = \lambda(E_{kk})$, $1 \leq k \leq n$ so that $\lambda(h_{km}) = \lambda_k - \lambda_m$ for all $(k, m) \in I$. Under the adjoint action of $\mathfrak{h}$, $e_{km}$ has weight $\alpha_{km} = E_{kk}^* - E_{mm}^*$ and $f_{km}$ has weight $-\alpha_{km}$. Define $\rho = \frac{1}{2} \sum_k \alpha_{km}$. Then $\rho(h_{km}) = m - k$ for all $(k, m) \in I$.

$R^+ = \{\alpha_{km} : (k, m) \in I\}$

$R = R^+ \cup (-R^+)$

$P = \{\lambda \in \mathfrak{h}^* : \lambda_k - \lambda_m \in \mathbb{Z}, \text{ for all } (k, m) \in I\}$

$R^+(\lambda) = \{\alpha_{km} \in R^+ : \lambda_k - \lambda_m + m - k \in \mathbb{N}\}$

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are the respective definitions of the positive roots, roots, integral weights and relative positive roots where \( \mathbb{N} \) denotes the positive integers. The Weyl group \( W \) is generated by the reflections \( s_{km} \) which act on \( \mathfrak{h}^* \) by the rule \( (s_{km})_q = \lambda_{(k,m)}q \) where \( (k,m) \) is the transposition and \( 1 \leq q \leq n \). If \( w \in W \) then \( w \cdot \lambda = w(\lambda + \rho) - \rho \).

\( U(\mathfrak{g}) \) is the enveloping algebra of a Lie algebra \( \mathfrak{g} \). For technical reasons a particular Kostant basis for \( U(\mathfrak{g}) \) will be used throughout. Arrange \( I \) and \( S \) in lexicographic order with the second index taken first. This induces an ordering of multi-indices (lexicographic over \( I \) or \( S \)). Let \( \mathbb{Z}^+ = \mathbb{N} \cup \{0\} \). If \( \nu \in (\mathbb{Z}^+)^I \), define \( f_{\nu} = \prod_{I} f_{km,\nu(km)} \), where the product is taken strictly in order and \( f_{km,q} = q!^{-1} f_{km}^q \). \( e_{\nu} \) is then defined under the adjoint action of \( \mathfrak{h} \) on \( U(\mathfrak{g}) \) is \( p(\nu) = \sum_{I} \nu_{km} \alpha_{km} \). If \( \sigma \in (\mathbb{Z}^+)^S \) set \( h_{\sigma} = \prod_{S} (h_{kk+1})^{\alpha_{kk+1}} \).

The set of monomials of the form \( f_{\nu} h_{\sigma} e_{\nu} \) is a Kostant basis for \( U(\mathfrak{g}) \). Let \( \mathbb{Z}U(\mathfrak{g}) \) be the ring of integral combinations of basis monomials and let \( \mathbb{Z}\mathfrak{g} \) be the corresponding Lie ring of elements in \( \mathbb{Z}U(\mathfrak{g}) \) of degree one. For any integral domain \( R \) let \( RU(\mathfrak{g}) = R \otimes_{\mathbb{Z}} \mathbb{Z}U(\mathfrak{g}) \) and \( R\mathfrak{g} = R \otimes_{\mathbb{Z}} \mathbb{Z}\mathfrak{g} \). There are analogous definitions for \( \mathfrak{b} \), \( \mathfrak{n} \), \( \mathfrak{n}^* \), and \( \mathfrak{h} \) using the appropriate subset of the standard basis.

Let \( \pi : RU(\mathfrak{g}) \to RU(\mathfrak{h}) \) be the projection induced by the decomposition
\[
RU(\mathfrak{g}) = (R\mathfrak{n} RU(\mathfrak{g}) + RU(\mathfrak{g}) R\mathfrak{n}^*) \oplus RU(\mathfrak{h}).
\]

Also if \( X \in RU(\mathfrak{b}) \) and \( \mu \in R\mathfrak{h}^* = R \otimes P \) define \( X(\mu) = 1 \otimes \mu(X) \) according to the decomposition \( RU(\mathfrak{b}) = RU(\mathfrak{n}) \otimes_R RU(\mathfrak{h}) \). Given \( X, Y \in RU(\mathfrak{g}) \) define the contravariant form \( (X,Y) = \pi((X,Y)) \). If \( \mu \in R\mathfrak{h}^* \) define \( (X,Y)_\mu = \pi((X,Y))(\mu) \). (See [5] for the properties of contravariant forms.)

The Jantzen filtration of \( U(\mathfrak{n}) \) with respect to \( \lambda \in \mathfrak{h}^* \) is defined as follows. Let \( A \) be the localization of \( K[t] \) at the principal ideal generated by \( t \). If \( q \in \mathbb{Z}^+ \) then let
\[
AU(\mathfrak{n})^q = \{ X \in AU(\mathfrak{n}) : v_t(X,Y)_{\lambda+t\rho} \geq q, \text{for all } Y \in AU(\mathfrak{n}) \}
\]
where \( v_t \) denotes valuation by powers of \( t \). Let \( AU(\mathfrak{n})_q = AU(\mathfrak{n})^q \setminus AU(\mathfrak{n})^{q+1} \). Then define \( U(\mathfrak{n})^q = \varepsilon \otimes 1(AU(\mathfrak{n})^q) \), where \( \varepsilon \) is the evaluation homomorphism of \( A \) at \( t = 0 \). The Jantzen filtration of a Verma module \( M(\lambda) \) with highest weight \( \lambda \) is the filtration \( M(\lambda)^q = U(\mathfrak{n})^q \cdot v \) where \( v \) is the highest weight vector.

1. Introduction

Suppose that \( 1 \leq i < j \leq n \) and \( d \in \mathbb{N} \). Let \( \Lambda^d_{ij} \) be the root hyperplane corresponding to \( -d \delta_{ij} \), that is, if \( \lambda \in \Lambda^d_{ij} \), \( \lambda_i - \lambda_j + j - i = d \).

A Shapovalov element is an element \( Y_{ij,d} \in AU(\mathfrak{b}) \) of weight \( -d \delta_{ij} \) chosen so that \( Y_{ij,d} \cdot v \) is a primitive vector whenever \( v \) is a primitive vector of weight \( \lambda \in \Lambda^d_{ij} \). Such an element induces a containment of Verma modules \( M(s_{ij} \cdot \lambda) \supseteq M(\lambda) \). Shapovalov elements are not unique but \( Y_{ij,d}(\lambda) \in U(\mathfrak{n}) \setminus \{0\} \) is unique up to a non-zero scalar multiple. (For details see [5] or [3].)
The Jantzen conjecture relates the Jantzen filtration of $M(s_{ij} \cdot \lambda)$ to that of $M(\lambda)$.

$$M(s_{ij} \cdot \lambda)^q = M(s_{ij} \cdot \lambda) \cap M(\lambda)^{q+N},$$

where $N = |R^+(\lambda)| - |R^+(s_{ij} \cdot \lambda)|$. The Jantzen conjecture was settled positively in the early eighties as a consequence of the Kazhdan-Lusztig conjecture. The solution was, for this reason, very indirect. The main thrust of this paper is to attempt a direct approach.

When $q = 0$, the Jantzen conjecture implies that $Y_{ij,d}(\lambda) \in U(n)_N$. The preliminary question here is whether a Shapovalov element can be found which satisfies $Y_{ij,d}(\lambda + tp) \in AU(n)^N$ for all $\lambda \in \Lambda^d_{ij}$. The general answer is that there is no such element. This is proved for several low-rank examples ($\S$ 4; $\Lambda^1_{14}$, $\Lambda^2_{14}$, and $\Lambda^1_{15}$) and is conjectured if $j - i > 2$.

In fact the same is true even if the condition is weakened to allow local Shapovalov elements. A local Shapovalov element is an element $\hat{Y} \in ZU(b)$ satisfying the condition $Y(\lambda) \in KY_{ij,d}(\lambda)$ for all $\lambda \in \Lambda^d_{ij}$. More precisely the question becomes whether there is a local Shapovalov element $\hat{Y} \in ZU(b)$ satisfying the conditions,

$$Y(\lambda + tp) \in AU(n)^N$$

$$Y(\lambda) \neq 0$$

The answer depends strongly on $\lambda$.

A singular weight is defined to be a weight, $\lambda_0$, where it is impossible for any local Shapovalov element to satisfy (1.1) and (1.2) in a (Zariski) open neighborhood of $\lambda_0$. The existence of singular weights in each of the low-rank cases listed above shows that there is no canonical choice of Shapovalov element compatible with the Jantzen conjecture.

On the positive side, Carter’s theory of lowering operators [1] can be brought to bear on the problem. The main idea is to construct a series of elements whose projections in $ZU(b)(\lambda + tp)$ are orthogonal to the standard Kostant basis of $AU(n)$ ($\S$ 2). In the examples of $\S$ 4 each element of the series is a local Shapovalov element that satisfies (1.1) and (1.2) for $\lambda$ in an open set of weights. These sets form an open covering of the non-singular weights in each example. A series with these properties is called a local system of Shapovalov elements.

In generalizing these results there are three distinct conjectures. The first (conjecture 2.3(i)) gives a formula for a common denominator over $K(t)U(n)$. The second (conjecture 2.3(ii)) is the assertion that the construction described below always produces local Shapovalov elements. The third (conjecture 3.10) is the determination that this set of elements forms a local system of Shapovalov elements. Based on these conjectures the singular weights are determined by a simple condition on the weight (proposition 3.11).
2. Results on Lowering Operators

Carter and Lusztig [2] introduced the element $S_{ij} \in \mathbb{Z}U(b)$ as the determinant of

$$
\begin{pmatrix}
(f_{i+1} & -(h_{i+1} + 1) & 0 & 0 & \cdots & 0 \\
(f_{i+2} & f_{i+1} + 2 & -(h_{i+2} + 2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \iddots & \vdots \\
(f_{j-1} & f_{j-1} + 1 & \cdots & f_{j-2} & -(h_{j-1} + j - i - 1) \\
(f_{ij} & f_{i+1} & \cdots & f_{j-2} & f_{j-1})
\end{pmatrix}
$$

where the products are taken in column order. Carter and Lusztig proved that the element $(S_{ij})^d$ is a Shapovalov element.

More recently Carter [1] used these elements to define lowering operators. If $\nu \in (\mathbb{Z}^+)^I$ is a multi-index define

$$S_\nu = \prod_I (S_{km})^{\nu(km)}$$

where the product is taken in the fixed order of $I$. The main results are that these are orthogonal with respect to the contravariant form and that the norm squares are given by the following formula. Let $\frac{\nu}{q} = q! \frac{\nu}{q}$.

**Proposition 2.1.** [1, Theorem 9] Let $\nu \in (\mathbb{Z}^+)^I$ be a multi-index. If $k < l \leq m$ define $h(klm) = h_{kl} + l - k - 1 + (\nu(l, m+1) + \cdots + \nu(lm)) - (\nu(k, m+1) + \cdots + \nu(kn))$. Then

$$(S_\nu, S_\nu) = \prod_I \nu(km)! \left[ \frac{h(klm)}{\nu(km)} \prod_{k < l < m} \left[ \frac{h(klm)}{\nu(km)} \right] \left[ \frac{h(klm) + \nu(lm) + 1}{\nu(km)} \right] \right].$$

The next result shows how these Shapovalov elements are naturally connected to the shift parameter $N$.

**Proposition 2.2.** Suppose that $\lambda \in \Lambda^d_{ij}$ and let $N = |R^+(\lambda)| - |R^+(s_{ij} \cdot \lambda)|$. Then $
u_l (S^d_{ij}, S^d_{ij})_{\lambda + t \rho} = N$.

**Proof.** According to proposition 2.1, $(S^d_{ij}, S^d_{ij})_{\lambda + t \rho}$ is given by the formula

$$d! \left[ \frac{(j-i)t + d - 1}{d} \prod_{i < l < j} \left[ \frac{(l-i)t + \lambda_l - \lambda_l + l - i - 1}{d} \right] \left[ \frac{(l-i)t + \lambda_l + l - i}{d} \right] \right].$$

$N$ is the cardinality of the set $(R^+(\lambda) \setminus R^+(s_{ij} \cdot \lambda)) \cap s_{ij}(-R^+)$, that is, the number of pairs, $(k, m) \in I$, so that $(ij)k > (ij)m$, $\lambda_k - \lambda_m + m - k \in \mathbb{N}$.
and \( \lambda_{(ij)k} - \lambda_{(ij)m} + (ij)m - (ij)k \not\in \mathbb{N} \). These conditions break down into three cases corresponding to the factors.

The leading term has one factor of \( t \) and corresponds to the case \((k, m) = (i, j)\). If \( k = i \) and \( i < m < j \) then \( \lambda_i - \lambda_m + m - i = d + (\lambda_j - \lambda_m + m - j) \in \{1, \ldots, d\} \) if and only if \( t \) is a factor of the first term in the product. If \( i < k < j \) and \( m = j \) then \( \lambda_i - \lambda_k + k - i = d + (\lambda_j - \lambda_k + k - j) \in \{0, \ldots, d - 1\} \) if and only if \( t \) is a factor of the second term in the product. \( \square \)

Before constructing a series of elements using the lowering operators, we need to summarize some notation and results on multi-indices and weights.

If \( \nu \in (\mathbb{Z}^+)^I \) is a multi-index, let \( \nu(k) = \sum_m \nu(k, m) \). Recall that \( p(\nu) = \sum_m \nu_{km} \alpha_{km} \). Define \( \nu_{\min} = \min\{\mu : p(\mu) = p(\nu)\} \) and define \( \nu_{\max} = \max\{\mu : p(\mu) = p(\nu)\} \) where the maximum and minimum are determined relative to the fixed order of \( I \).

Suppose that \( p(\nu) = d\alpha_{ij} \). Then \( \nu_{km} = 0 \) unless \( i \leq k < m \leq j \). Moreover \( \nu(i) = d \) and \( \nu(k) \in \{0, 1, \ldots, d\} \) for \( i < k < j \). For example, \( \nu_{\max}(k) = d\delta_{ki} \) while \( \nu_{\min}(k) = d \) if \( i \leq k < j \) and 0 otherwise. Also \( h(klm) = h_{kl} + l - k - 1, k < l \leq m \).

Given \( \nu_i = d \) and \( \nu_k \in \{0, 1, \ldots, d\} \) for \( i < k < j \), there is a multi-index \( \nu \) with \( p(\nu) = d\alpha_{ij} \) so that \( \nu(k) = \nu_k, i \leq k < j \). To construct \( \nu \) define for \( J \subseteq [i, j] \subseteq \mathbb{N} \), \( M J = \max\{\nu_k : k \in J\} \) and \( M \emptyset = 0 \). Set \( \nu_j = d \). Then take \( \nu_{km} = -M[k, m] + M[k, m] + M(k, m) - M(k, m) \).

Define \( \widehat{A}_{\mu
u} \in \mathbb{Z}U(b) \) to be the coefficient of \( f_\mu \) in \( S_\nu \) and let \( A_{\mu
u} = \widehat{A}_{\mu
u}(\lambda + t\rho) \). If \( A_{\mu
u} \neq 0 \) then \( p(\mu) = p(\nu), \mu \leq \nu, \) and \( \mu(k) \geq \nu(k) \) for all \( k \). In some cases there are known formulae for \( A_{\mu
u} \). In particular, when \( p(\nu) = d\alpha_{ij} \), \( S_{\nu_{\max}} = S_{ij}^d \) and

\[
\widehat{A}_{\nu_{\max}} = d! \prod_{i < k < j} \nu(k)! \left[ \frac{h_{ik} + k - i}{d - \nu(k)} \right].
\]

Also \( \widehat{A}_{\nu_{\min}} = (d!)^{j-i} \) [2].

The contravariant form with respect to \( \lambda + t\rho \) is non-degenerate on \( K(t)U(n) \) because the corresponding Verma module is irreducible. If \( p(\nu) = d\alpha_{ij} \), define \( X_\nu \) to be the unique element of \( K(t)U(n) \) satisfying \((f_\mu, X_\nu)(\lambda + t\rho) = \delta_{\mu\nu} \). The orthogonality property of lowering operators leads to an explicit formula for \( X_\nu \), that is,

\[
X_\nu = \sum_{\mu \geq \nu} s^{-1}_{\mu} A_{\nu\mu} S_\mu(\lambda + t\rho),
\]

where \( s_{\mu} = (S_\mu, S_\mu)_{\lambda + t\rho} \). With this description, \( q_\nu \), the lowest common denominator of \( X_\nu \) with respect to the Kostant basis of \( K[t]U(b)(\lambda + t\rho) \), can be determined by direct calculation. By construction \( q_\nu X_\nu \in AU(n)_M \) where \( M = v_t q_\nu \). Due to the nature of the coefficients, it is always possible to choose an element \( Y_\nu \in \mathbb{Z}U(b) \) so
that $Y_\nu(\lambda + t\rho) = q_\nu X_\nu$. In this way $Y_\nu(\lambda) \in U(n)^M$. $Y_\nu$ is not itself determined uniquely by the construction but the projection $Y_\nu(\lambda + t\rho)$ is unique.

The determination of a general formula for $q_\nu$ is the next step. In the examples the calculation of each $q_\nu$ was difficult, but a relatively simple formula for $q_\nu$ became evident. Moreover in each example this construction had a surprising property. $Y_\nu$ was always a local Shapovalov element. These properties provide a clear line of attack on the problem. The following conjectures combine these observations.

**Conjecture 2.3.** Suppose that $\lambda \in \Lambda^d_{ij}$ and let $p(\nu) = d\alpha_{ij}$.

(i) If $q_\nu = \prod_l \left[ (m-k)t + \lambda_k - \lambda_m + m - k - 1 \right]_{\nu(k)}$ then $q_\nu X_\nu \in K[t]U(n)$.

(ii) $Y_\nu$ is a local Shapovalov element.

When $\nu = \nu_{\max}$,

$$Y_{\nu_{\max}}(\lambda + t\rho) = q_{\nu_{\max}}^{-1} A_{\nu_{\max}} \nu_{\max} \cdot S_{ij}^d(\lambda + t\rho) = S_{ij}^d(\lambda + t\rho),$$

and the conjectures hold trivially. $Y_{\nu_{\max}}$ will be identified with $S_{ij}^d$.

3. LOCAL SYSTEMS AND SINGULAR WEIGHTS

In this section results that depend on Conjecture 2.3 will be indicated by $\dagger$. Also we will make a standing assumption that for any multi-index $\nu$, $p(\nu) = d\alpha_{ij}$ and that for any weight $\lambda$, $\lambda \in \Lambda^d_{ij}$. Recall that $N = |R^+\nu(\lambda)| - |R^+\nu(s_{ij} \cdot \lambda)|$.

**Definition 3.1.** $X \in ZU(b)$ is canonical for $\lambda$ if $X(\lambda + t\rho) \in AU(n)^N$ and $X(\lambda) = C S_{ij}^d(\lambda)$ where $C \in K \setminus \{0\}$. If there is no element $X \in ZU(b)$ that is canonical for an open set of weights containing $\lambda$ then $\lambda$ is a singular weight.

As a first application of this approach we can quantify exactly when Carter’s Shapovalov element $S_{ij}^d$ is canonical.

**Proposition 3.2.** $S_{ij}^d \in AU(n)_{N - \delta}$ where $\delta = v_t(A_{\nu_{\max}} \nu_{\max})$.

**Proof.** $S_{ij}^d = Y_{\nu_{\max}}$. Since $s_{\nu_{\max}} = q_{\nu_{\max}} A_{\nu_{\max}} \nu_{\max}$, $v_t(q_{\nu_{\max}}) = v_t(s_{\nu_{\max}}) - v_t(A_{\nu_{\max}} \nu_{\max})$.

But by proposition 2.2, $v_t(s_{\nu_{\max}}) = N$. \( \square \)

**Corollary 3.3.** $S_{ij}^d$ is canonical for $\lambda$ if and only if $\widehat{A_{\nu_{\max}}} \nu_{\max}(\lambda) \neq 0$.

These results do generalize to the other elements of the series via conjecture 2.3. However $Y_\nu(\lambda)$ is generally a multiple of $S_{ij}^d(\lambda)$. Let $C_\nu(\lambda) \in K$ be the multiplier. The following results will provide a formula for $C_\nu(\lambda)$.

**Definition 3.4.** $D_\nu = \prod_{i < k < j} (-1)^{\nu(k)} \nu(k)! \prod_{k < m < j} \left[ h_{km} + m - k - 1 \right]_{\nu(k)}$. 
Lemma 3.5. Let $Z_\nu = q_\nu \left( s_{\nu_{\max}}^{-1} A_{\nu \nu_{\max}} \right)$. Then $v_t(Z_\nu) = v_t(D_\nu(\lambda + t\rho))$.

Proof. Because $Z_{\nu_{\max}} = 1$ we have $Z_\nu \left( \frac{A_{\nu_{\max} \nu_{\max}}}{A_{\nu \nu_{\max}}} \right) = \left( \frac{q_\nu}{q_{\nu_{\max}}} \right)$. Using the formulae above,

\[
\frac{q_\nu}{q_{\nu_{\max}}} = \prod_{i < k < m \leq j} \left[ (m - k)t + \lambda_k - \lambda_m + m - k - 1 \right]^{\nu(k)}
\]

\[
= D_\nu(\lambda + t\rho) \prod_{i < k < j} (-1)^{\nu(k)} (\nu(k))^{-1} \left[ (j - k)t + \lambda_k - \lambda_j + j - k - 1 \right]^{\nu(k)}
\]

\[
= D_\nu(\lambda + t\rho) \prod_{i < k < j} (\nu(k))^{-1} \left[ -(j - k)t + \lambda_i - \lambda_k + k - i - d + \nu(k) \right]^{\nu(k)}
\]

because $\left[ \frac{x}{q} \right] = (-1)^q \left[ \frac{x + q - 1}{q} \right]$. On the other hand, since $\left[ \frac{x}{q} \right] = \left[ \frac{x}{q - r} \right] \left[ \frac{x - q + r}{r} \right]$, $A_{\nu_{\max} \nu_{\max}} = \prod_{i < k < j} (\nu(k))^{-1} \left[ (k - i)t + \lambda_i - \lambda_k + k - i - d + \nu(k) \right]^{\nu(k)}$.

Comparing the two equations, $v_t(Z_\nu) = v_t(D_\nu(\lambda + t\rho))$. \qed

Proposition 3.6 (†). $C_\nu(\lambda) = D_\nu(\lambda)$.

Proof. The coefficient of $f_{\nu_{\max}}$ in $Y_\nu(\lambda + t\rho)$ is $Z_\nu A_{\nu_{\max} \nu_{\max}}$. By the proof of 3.5, $\varepsilon(Z_\nu) = D_\nu(\lambda)$, provided that $A_{\nu_{\max} \nu_{\max}}(\lambda) \neq 0$. So the coefficient of $f_{\nu_{\max}}$ in $Y_\nu(\lambda)$ is $D_\nu(\lambda) A_{\nu_{\max} \nu_{\max}}(\lambda)$. However, according to conjecture 2.3, the same coefficient is given by $C_\nu(\lambda) A_{\nu_{\max} \nu_{\max}}(\lambda)$. Since these are equal for all $\lambda \in \Lambda^+_{ij}$, $C_\nu(\lambda) = D_\nu(\lambda)$ \qed

Now 3.2 and 3.3 can be generalized to any $Y_\nu$.

Proposition 3.7 (†). If $C_\nu(\lambda) \neq 0$ then $Y_\nu(\lambda + t\rho) \in AU(n)_{N - \delta}$ where $\delta = v_t(A_{\nu_{\max}})$.

Proof. By the definition of $Z_\nu$, $v_t(Z_\nu) = v_t(q_\nu) + v_t(A_{\nu \nu_{\max}}) - v_t(s_{\nu_{\max}})$. Applying 3.4 and 2.1, $v_t(D_\nu(\lambda + t\rho)) = v_t(q_\nu) + \delta - N$. If $D_\nu(\lambda) \neq 0$ then $v_t(q_\nu) = N - \delta$. \qed

Corollary 3.8 (†). Let $y_\nu = D_\nu \hat{A}_{\nu \nu_{\max}}$. $Y_\nu$ is canonical for $\lambda$ if, and only if, $y_\nu(\lambda) \neq 0$.

The existence of singular weights forces us to consider local Shapovalov elements. On the basis of conjecture 2.3 we obtain elements that are canonical for specified open sets of weights. The singular weights must lie outside the union of these sets. However, in every example of §4 the singular weights actually dovetail with the union, that is, if $y_\nu(\lambda) = 0$ for all $\nu$ with weight $p(\nu) = d\alpha_{ij}$ then $\lambda$ is singular. It seems then that the best description of the dependence on $\lambda$ in the Jantzen conjecture is embodied in the following definition and conjecture.
Definition 3.9. A local system of Shapovalov elements is a set, \( \{T_k : 1 \leq k \leq m\} \), where each \( T_k \in \mathbb{Z}U(b) \) has an associated \( g_k \in \mathbb{Z}U(b) \) that satisfies the following conditions.

(i) \( T_k \) is canonical for all \( \lambda \) with \( g_k(\lambda) \neq 0 \).

(ii) If \( g_k(\lambda) = 0 \) for all \( k \) then \( \lambda \) is a singular weight.

Conjecture 3.10. \( \{Y_{\nu} : p(\nu) = d\alpha_{ij}\} \) is a local system of Shapovalov elements.

It follows from Conjecture 2.3 that (i) is satisfied. It would suffice to show that any weight \( \lambda \) satisfying \( y_{\nu}(\lambda) = 0 \) for all \( \nu \) with \( p(\nu) = d\alpha_{ij} \) is singular. That has been verified in each example below. The examples also make clear that although there is one \( Y_{\nu} \) for each \( \nu \) satisfying \( p(\nu) = d\alpha_{ij} \) only a subset of these (representing a base of the open covering) is needed.

A local system characterizes the singular weights. What can be said about the predicted patterns of singular weights? The next result would provide a necessary and sufficient condition for singularity. Recall that \( \mathbb{N} = \mathbb{Z}^+ \setminus \{0\} \).

Proposition 3.11. \( y_{\nu}(\lambda) = 0 \) for all \( \nu \) with \( p(\nu) = d\alpha_{ij} \) if, and only if, there is a pair \( (k, m) \) with \( i < k < m < j \) so that

(i) \( \lambda_i - \lambda_k + k - i \in \mathbb{Z}^+ \)

(ii) \( \lambda_k - \lambda_m + m - k \in \mathbb{N} \), and

(iii) \( \lambda_m - \lambda_j + j - m \in \mathbb{Z}^+ \).

Proof. Suppose that \( \lambda \) satisfies the conditions for the pair \( (k, m) \) and that \( y_{\nu}(\lambda) \neq 0 \). Then referring to the formulae, \( [\lambda_k - \lambda_m + m - k - 1]_{\nu(k)} \neq 0 \) and \( [\lambda_i - \lambda_k + k - i]_{d-\nu(k)} \neq 0 \). If \( \nu(k) > 0 \), then \( \lambda_k - \lambda_m + m - k \not\in \{1, 2, \ldots, \nu(k)\} \). Due to (ii) this implies that \( \lambda_k - \lambda_m + m - k > \nu(k) \). If \( \nu(k) = 0 \), the inequality remains true. By similar reasoning, \( \lambda_i - \lambda_k + k - i > d - \nu(k) \). Adding the inequalities yields that \( \lambda_i - \lambda_m + m - i > d - 1 \). Condition (iii) then implies that \( \lambda_i - \lambda_m + m - i = d \). But this contradicts the first inequality we obtained since \( \lambda_k - \lambda_m + m - k = d - (\lambda_i - \lambda_k + k - i) < \nu(k) + 1 \).

Suppose that \( \lambda \) does not satisfy the conditions, that is, there is no pair \( (k, m) \) satisfying (i)-(iii). It will suffice to construct a multi-index \( \nu \) so that \( y_{\nu}(\lambda) \neq 0 \). We need only specify the row sums. Choose \( \nu \) so that \( p(\nu) = d\alpha_{ij}, \nu(j - 1) = d \), and \( \nu(k) \in \{0, 1, \ldots, d\} \) is minimal satisfying \( [\lambda_i - \lambda_k + k - i]_{d-\nu(k)} \neq 0 \) for each \( k, i < k < j - 1 \).

By this choice, \( \hat{A}_{\nu_{\max}} \neq 0 \). It remains to show that \( D_{\nu}(\lambda) \neq 0 \). Suppose to the contrary that there is a pair \( (k, m) \) with \( i < k < m < j \) and \( [\lambda_k - \lambda_m + m - k - 1]_{\nu(k)} = 0 \). Then \( \nu(k) > 0 \) and \( \lambda_k - \lambda_m + m - k \in \{1, 2, \ldots, \nu(k)\} \). By the minimality of \( \nu(k) \), \( [\lambda_k - \lambda_m + m - k - 1]_{d-\nu(k)-1} = 0 \), hence \( \lambda_k - \lambda_m + m - i \in \{0, 1, \ldots, d - \nu(k)\} \). But then \( \lambda_m - \lambda_j + j - m = d - (\lambda_i - \lambda_m + m - i) \in \{0, 1, \ldots, d - 1\} \), \( (k, m) \) satisfies the conditions. So for each pair \( (k, m) \) with \( i < k < m < j \), \( [\lambda_k - \lambda_m + m - k - 1]_{\nu(k)} \neq 0 \) so that \( D_{\nu}(\lambda) \neq 0 \). \( \square \)
This result has some nice properties that can be viewed as indirect evidence for the conjectures. For example, the condition is independent of $d$. There are no singular weights if $j - i \leq 2$. When $j - i > 2$ there are some interesting regularities. If $\lambda$ is a regular dominant integral weight then $\lambda$ is singular. If $\lambda$ is dominant integral and non-singular then $\lambda$ lies along one edge of the dominant chamber in the root wall corresponding to the root $\alpha_{i+1,j-1}$. In this case, $S_{ij}^d$ is canonical for $\lambda$.

4. EXAMPLES

In each example, a local system is presented and the calculation of the singular weights is summarized. Each conjecture has been verified in the following examples. The actual calculations required to generate the examples and verify the conjectures ran to about a ream of paper. In order to generalize these examples to $\Lambda_{ij}^d$ refer to $\Lambda_{ik}^d$, where $k = j - i + 1$, and make appropriate index changes. Note that $Y_{\nu}$ is canonical for $\lambda$ if and only if the coefficient of $f_\nu$ in $Y_\nu(\lambda)$ is non-zero.

To simplify the notation a subscript of $q$ in place of a multi-index refers to the $q$th multi-index with weight $d\omega_{ij}$ in the fixed order of multi-indices. In reference to weights we will use the notation $\xi = \lambda_1 - \lambda_2$, $\zeta = \lambda_2 - \lambda_3$, and $\psi = \lambda_3 - \lambda_4$. Also we will use the in-line notation $[x,q]$ for $[x]_q$.

Example ($\Lambda_{12}^d$). $Y_0 = S_{12}^d = f_{12}^d$ and $y_0 = d!$. So there are no singular weights and $\{Y_0\}$ is a local system.

Example ($\Lambda_{13}^d$). In this case, $f_k = f_{12,d-k} f_{13,k} f_{23,d-k}$ for $0 \leq k \leq d$.

$$Y_0 = d!^2 \sum_{k=0}^{d} f_k \begin{bmatrix} h_{13} - d + 1 \\ d - k \end{bmatrix} \begin{bmatrix} h_{23} - d + k \\ k \end{bmatrix}$$

and $y_0 = (-1)^d d!^3$. Again there are no singular weights, $Y_0$ is a Shapovalov element and $\{Y_0\}$ is a local system.

Example ($\Lambda_{14}^d$). Here $f_0 = f_{12} f_{23} f_{34}$, $f_1 = f_{12} f_{24}$, $f_2 = f_{13} f_{34}$ and $f_3 = f_{14}$. In the table below are listed the coefficients of each $Y_\nu$. For example, $Y_2 = f_0 (h_{14} + 1) + f_1 h_{34} + f_2 (h_{12} + 1)(h_{14} + 1) + f_3 (h_{12} + 1) h_{34}$.

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<tr>
<th></th>
<th>$Y_0$</th>
<th>$Y_2$</th>
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<tbody>
<tr>
<td>$f_0$</td>
<td>$h_{23} + h_{13}(h_{14} + 2)h_{24}$</td>
<td>$h_{14} + 1$</td>
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<tr>
<td>$f_1$</td>
<td>$h_{23}(h_{14} + 1)h_{34} + h_{12}(h_{14} + 2)h_{34}$</td>
<td>$h_{34}$</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$h_{23}(h_{14} + 1)(h_{24} + 1)$</td>
<td>$(h_{12} + 1)(h_{14} + 1)$</td>
</tr>
<tr>
<td>$f_3$</td>
<td>$h_{23}(h_{24} + 1)h_{34}$</td>
<td>$(h_{12} + 1)h_{34}$</td>
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In this case we have $y_0 = h_{23}$ while $y_2 = -(h_{12} + 1)$. To show that $\{Y_0, Y_2\}$ is a local system we must verify that that any weight $\lambda$ satisfying $\xi = -1$ and $\zeta = 0$ is singular.
Let $x_k \in K[t]$ be the coefficient of $f_k$ in $X(\lambda + t\rho)$ where $X \in ZU(n)$ is canonical for an open set of weights containing $\lambda$. Let $x_k^m = x_k^m(\xi, \zeta)$ be the coefficient of $t^m$ in $x_k$. Calculation of $K[t]U(n)^N$ yields relations between various $x_k^m$. In particular we have,

$$-(\zeta + 1)x_1^1 + x_1^1 = -x_0^0 \quad \text{if } \xi = -1, \text{ while}$$

$$-x_1^1 + x_1^1 = 2x_0^1 \quad \text{if } \xi + \zeta = -1.$$

Since both relations hold when $\xi = -1$ and $\zeta = 0$, $x_0^0(-1, 0) = 0$. So $X(\lambda)$ cannot be a non-zero multiple of $S_{14}(\lambda)$.

**Example** ($\Lambda_{14}^0$). Using the same format as before,

$$f_0 = f_{12,2} f_{23,2} f_{34,2} \quad f_1 = f_{12,2} f_{23} f_{24} f_{34}$$

$$f_2 = f_{12,2} f_{24,2} \quad f_3 = f_{12} f_{13} f_{23} f_{34,2}$$

$$f_4 = f_{12} f_{13} f_{24} f_{34} \quad f_5 = f_{12} f_{23} f_{14} f_{34}$$

$$f_6 = f_{12} f_{14} f_{24} \quad f_7 = f_{13,2} f_{34,2}$$

$$f_8 = f_{13} f_{14} f_{34} \quad f_9 = f_{14,2}$$

And the tables of coefficients are:

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<th>$Y_0$</th>
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<th>$Y_4$</th>
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<th>$Y_{11}$</th>
<th>$Y_{12}$</th>
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<th>$Y_{14}$</th>
<th>$Y_{15}$</th>
<th>$Y_{16}$</th>
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<tr>
<td>$f_0$</td>
<td>$32[h_{23},2] + 8(h_{13} - 1)(h_{14} + 1)(h_{24} - 1) {(h_{13} - 2)(h_{14} + 2)(h_{24} - 2) + 8(h_{23} - 1)}$</td>
<td>$f_1$</td>
<td>$16[h_{23},2]h_{14}(h_{34} - 1)$</td>
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<td>$f_3$</td>
<td>$16[h_{23},2]h_{14}h_{24} + 8(h_{13} - 1)(h_{23} - 1)[h_{14},1,2][h_{24},2]$</td>
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<td>$f_4$</td>
<td>$8[h_{23},2]h_{24}(h_{34} - 1) + 8(h_{23} - 1)(h_{14} + 1)h_{24}(h_{34} - 1) {(h_{12} - 1)h_{14} + h_{23}(h_{14} - 1)}$</td>
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<td>$f_5$</td>
<td>$8[h_{23},2]h_{24}(h_{34} - 1) + 8(h_{13} - 1)(h_{23} - 1)(h_{14} + 1)h_{24}(h_{34} - 1)$</td>
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<td>$f_6$</td>
<td>$8[h_{23},2]h_{14}h_{24} + 8(h_{12} - 1)(h_{23} - 1)(h_{14} + 1)h_{24}[h_{34},2]$</td>
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<td>$8[h_{23},2][h_{14},2][h_{24},1,2]$</td>
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<td>$f_8$</td>
<td>$8[h_{23},2]h_{14} + 1,2[h_{34} - 1]$</td>
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<td>$f_9$</td>
<td>$8[h_{23},2][h_{24} + 1,2][h_{34},2]$</td>
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<td>f₀</td>
<td>16h₂₃h₁₄ + 8(h₁₃ - 1)[h₁₄ + 1, 2](h₂₄ - 1)</td>
<td>8[h₁₄, 2]</td>
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<td>f₁</td>
<td>8h₂₃(h₃₄ - 1) + 4(h₁₄ + 1)(h₃₄ - 1) { (h₁₂ - 1)h₁₃ + 2(h₁₃ - 1)h₂₄ }</td>
<td>8h₁₄(h₃₄ - 1)</td>
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<td>f₂</td>
<td>8h₂₃h₁₄[h₃₄, 2] + 8(h₁₂ - 1)(h₁₄ + 1)[h₃₄, 2]</td>
<td>8[h₃₄, 2]</td>
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<td>f₃</td>
<td>4h₂₃[h₃₄, 2]h₂₄ + 2[h₁₄ + 1, 2] { (h₁₂ + 1)h₂₃(h₁₄ - 1) + (h₁₂ + 2)(h₁₂ - 1)(h₃₄ - 1) }</td>
<td>4(h₁₂ + 1)[h₁₄, 2]</td>
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<td>f₄</td>
<td>4h₂₃h₁₄h₂₄(h₃₄ - 1) + 2[h₁₄ + 1, 2](h₃₄ - 1) { (h₁₂ - 1) + (h₁₂ - 1)(h₂₃ - 1) }</td>
<td>4(h₁₂ + 1)h₁₄(h₃₄ - 1)</td>
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<td>f₅</td>
<td>4h₂₃h₁₄h₂₄(h₃₄ - 1) + 2(h₁₂ + 1)(h₁₄ + 1)(h₃₄ - 1) { (h₁₂ - 1)h₃₄ + h₁₄h₂₃ }</td>
<td>4(h₁₂ + 1)h₁₄(h₃₄ - 1)</td>
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<td>f₆</td>
<td>4h₂₃h₂₄[h₃₄, 2] + 2(h₁₄ + 1)[h₃₄, 2] { (h₁₂ - 1) + (h₁₂ + 1)(h₂₃ - 1) }</td>
<td>4(h₁₂ + 1)[h₃₄, 2]</td>
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<td>f₇</td>
<td>4(h₁₂ + 1)h₂₃[h₃₄, 2](h₂₄ + 1)</td>
<td>4[h₁₂ + 1, 2]h₁₄, 2</td>
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<td>f₈</td>
<td>4(h₁₂ + 1)h₂₃h₄h₂₄ + 1(h₃₄ - 1)</td>
<td>4[h₁₂ + 1, 2]h₁₄(h₃₄ - 1)</td>
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Here we have

\[ y₀ = 32[h₂₃, 2] \]
\[ y₃ = -8(h₁₂ + 1)h₂₃ \]
\[ y₇ = 8[h₁₂ + 1, 2]. \]

All weights are non-singular except those \( \lambda \) with \( (\xi, \zeta) \in \{(-1, 0), (-1, 1), (0, 0) \} \). Arguing as in the previous example, the following relations show that these weights are singular.

\[ -x₀^1 + x₁^1 = 8x₀^0 \text{ if } \xi + \zeta = 0, \]
\[ -x₀^1 + 2x₁^1 = 4x₀^0 \text{ if } \xi + \zeta = -1, \]
\[ -(\xi + 1)x₀^1 + 2x₁^1 = -8x₀^0 \text{ if } \xi = -1, \]
\[ -x₀^1 + 2x₃^1 = -4x₀^0 \text{ if } \xi = 0, \text{ and} \]
\[ -(\xi + 1)x₀^1 + 2x₃^1 = 8x₀^0 \text{ if } \xi + \zeta = 0. \]

**Example (\( \Lambda_{15}^1 \)).**

\[ f₀ = f₁₂ f₂₃ f₃₄ f₄₅ \]
\[ f₁ = f₁₂ f₂₃ f₃₅ \]
\[ f₂ = f₁₂ f₂₄ f₄₅ \]
\[ f₃ = f₁₂ f₂₅ \]
\[ f₄ = f₁₃ f₃₄ f₄₅ \]
\[ f₅ = f₁₃ f₃₅ \]
\[ f₆ = f₁₄ f₄₅ \]
\[ f₇ = f₁₅ \]
<table>
<thead>
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<th>$Y_0$</th>
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</table>
| $f_0$ | $h_{23}(h_{24} + 1)h_{34}(h_{15} + 2) + (h_{15} + 3) \{(h_{12} h_{34} \{h_{24} + 1\} + (h_{13} + 1))\
\quad + (h_{25} + 2)h_{35} \{h_{12}(h_{14} + 1)h_{24} + h_{23}\}\} \ |
| $f_1$ | $h_{23}(h_{24} + 1)h_{34}h_{45} + (h_{15} + 3)h_{45} \{(h_{13} + 1)h_{23}(h_{14} + 1)(h_{25} + 2)\
\quad + h_{34}(h_{25} + 1) \{h_{12}(h_{14} + 2) + h_{23}(h_{14} + 1)\}\} \ |
| $f_2$ | $h_{23}(h_{24} + 1)h_{34}(h_{35} + 1)\
\quad + h_{34}(h_{15} + 3)(h_{25} + 1)(h_{35} + 1) \{h_{12}(h_{14} + 2) + h_{23}(h_{14} + 1)\}\} \ |
| $f_3$ | $h_{23}(h_{24} + 1)h_{34}(h_{15} + 2)(h_{35} + 1)h_{45}\
\quad + h_{12} h_{34}(h_{15} + 3)(h_{35} + 1)h_{45} \{(h_{13} + 1) + (h_{24} + 1)\} \ |
| $f_4$ | $h_{23}(h_{24} + 1)h_{34}(h_{25} + 2) + h_{23}(h_{14} + 1)(h_{24} + 1)(h_{15} + 3)(h_{25} + 2)h_{35} \ |
| $f_5$ | $h_{23}(h_{24} + 1)h_{34}(h_{15} + 2)(h_{25} + 2)h_{45} + (h_{13} + 1)h_{23}(h_{14} + 1)(h_{15} + 3)(h_{25} + 2)h_{45}\ |
| $f_6$ | $h_{23}(h_{24} + 1)h_{34}(h_{15} + 2)(h_{25} + 2)(h_{35} + 1)\ |
| $f_7$ | $h_{23}(h_{24} + 1)h_{34}(h_{25} + 2)(h_{35} + 1)h_{45}\ |

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<td>$h_{23}(h_{24} + 1)(h_{15} + 2)h_{45} + h_{12} h_{15} + 3)h_{45} {(h_{13} + 1) + (h_{24} + 1)} \</td>
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<tr>
<td>$f_2$</td>
<td>$(h_{13} + 2)h_{23}(h_{24} + 1) + (h_{15} + 3)(h_{25} + 1) {h_{12}(h_{24} + 2) + h_{23}(h_{23} + 2)(h_{14} + 1)}} \</td>
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<tr>
<td>$f_3$</td>
<td>$(h_{13} + 2)h_{23}(h_{24} + 1)(h_{15} + 2)h_{45} + h_{12} h_{15} + 3)h_{45} {(h_{13} + 2)h_{23} + (h_{14} + 2)} \</td>
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<td>$f_6$</td>
<td>$(h_{13} + 2)h_{23}(h_{24} + 1)(h_{15} + 2)(h_{25} + 2)\</td>
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<tr>
<td>$f_7$</td>
<td>$(h_{13} + 2)h_{23}(h_{24} + 1)(h_{25} + 2)h_{45}\</td>
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<tr>
<td>$f_1$</td>
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<td>$(h_{13} + 2)(h_{15} + 2)$</td>
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<tr>
<td>$f_3$</td>
<td>$(h_{12} + 1)h_{34} + (h_{12} + 1)(h_{14} + 1)(h_{15} + 3)h_{35}$</td>
<td>$(h_{12} + 1)(h_{15} + 2)$</td>
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<td>$f_4$</td>
<td>$(h_{12} + 1)h_{34}(h_{15} + 2)h_{45}$</td>
<td>$(h_{12} + 1)h_{45}$</td>
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<tr>
<td>$f_5$</td>
<td>$(h_{12} + 1)(h_{34} + (h_{12} + 1)(h_{14} + 1)(h_{15} + 3)h_{35}$</td>
<td>$(h_{12} + 1)(h_{13} + 2)(h_{15} + 2)$</td>
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<tr>
<td>$f_6$</td>
<td>$(h_{12} + 1)h_{34}(h_{15} + 2)(h_{35} + 1)$</td>
<td>$(h_{12} + 1)(h_{13} + 2)h_{45}$</td>
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<tr>
<td>$f_7$</td>
<td>$(h_{12} + 1)h_{34}(h_{35} + 1)h_{45}$</td>
<td>$(h_{12} + 1)(h_{13} + 2)h_{45}$</td>
</tr>
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</table>

In this case,

$y_2 = (h_{13} + 2)h_{23}(h_{24} + 1)$

$y_4 = (h_{12} + 1)h_{34}$

$y_6 = -(h_{12} + 1)(h_{13} + 2)$. 
All weights are non-singular except those \( \lambda \) with \((\xi, \zeta, \psi)\) satisfying one of the following conditions:

\[
\xi + 1 = 0 \quad \text{and} \quad \zeta = 0, \\
\xi + 1 = 0 \quad \text{and} \quad \zeta + \psi + 1 = 0, \quad \text{or} \\
\xi + \zeta + 2 = 0 \quad \text{and} \quad \psi = 0.
\]

The following relations show that these weights are singular.

\[
-x_0^1 + x_2^1 = 2x_0^0 \quad \text{if} \quad \xi + \zeta + 1 = 0, \\
-(\zeta + 1)x_0^1 + x_2^1 = -2x_0^0 \quad \text{if} \quad \xi + 1 = 0, \\
-(\zeta + \psi + 2)x_0^1 + x_1^1 = -x_0^0 \quad \text{if} \quad \xi + 1 = 0, \\
-x_0^1 + x_1^1 = 3x_0^0 \quad \text{if} \quad \xi + \zeta + \psi + 2 = 0, \quad \text{and} \\
-(\psi + 1)x_0^0 + x_1^1 = -x_0^0 \quad \text{if} \quad \xi + \zeta + 2 = 0.
\]

References


